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Risk aversion and Relationships in model-free *

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Abstract

This paper belongs to the study of decision making under risk. We will be interested in modeling the behavior of decision makers (hereafter referred to as DM) when they are facing risky choices. We first introduce both the general framework of decision making problem under risk and the different models of choice under risk that are well recognized in the literature. Then, we review different concepts of some increase in risk and risk aversion that are valid independently of any representation. We will introduce two new forms of behaviors under risk namely weak weak risk aversion and anti-monotone risk aversion. Note that the latter is related to anti-comonotony (a concept investigated in Abouda, Aouani and Chateauneuf (2008)) and represents a halfway between monotone and weak risk aversion. Finally, we discuss the relationships -in model-free- among some of these behaviors.

Keywords: Risk aversion, model-free concepts, Relationships, anti-comonotone, SMRA, MRA, ARA, WWRA.
JEL classification: D80-D81-G12.

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1 Introduction

nowadays, economic agents always make important and risky decisions: individuals make huge equity investments, companies make capital investments and design product lines, and farmers plant their crops, all without knowing what the future will bring.
Our work is organized as follows: We start by introducing the general framework of the decision making problem under risk. Then, we present different models of choice under risk. Finally, we give definitions of risk aversion and discuss the possible relationships among some of them in model free.

2 The decision problem under risk : general presentation

Risk is known as a particular case of uncertainty. So to deal with a decision problem under risk we have first to go by uncertainty. Most of our decisions are taken in uncertain situations: we have to choose a decision without knowing its consequences with certainty because they depend on events that may or may not occur. Decision theory was born to help decision makers take an optimal decision when they are facing a problem of risky choice. A decision problem under risk is usually described through a set S called the set of states of nature, identifying events with subsets of S.
During this work, we will focus, on a situation of risk. In this setting, we define the main properties of decision under risk, the different possible behaviors under risk and the relationships that can exist among some of them independently of any model. From now on, we will assume that the probability distribution on the set S is given; we are thus dealing with a problem of decision under risk (situations in which all events of S have "objective" probabilities with which the DM agrees). Indeed, the outcome of each de-

cision needs, first of all, an appropriate formalization of decision problem under risk.

3 Formalization and Notations

We suppose that we have a decision-maker faced with choices among risky assets X, the set V of such assets consisting of all bounded real random variables defined on a probability space $(S, \mathcal{A}, P)$[1] assumed to be sufficiently rich to generate any bounded real-valued random variable. S is the set of states of nature, $\mathcal{A}$ is a σ-algebra of events (i.e. of subsets of S), and P is a σ-additive non-atomic probability measure. Let V_0 containing only discrete elements of V.

Definition 3.1. *A family $\mathcal{A}$ of subsets of the universe S is called a sigma-algebra if it fulfils the three following properties:*

- $S \in \mathcal{A}$ *(i.e, S itself is an event);*
- $E \in \mathcal{A} \Rightarrow \overline{E} \in \mathcal{A}$ *(i.e, $\overline{E}$ is called the complement of the event E);*
- $E_1, E_2, E_3, ... \in \mathcal{A} \Rightarrow \bigcup_{i \geq 1} E_i \in \mathcal{A}$.

A risk can be described as an event that may or may not take place, and that brings about some adverse financial consequences. It is thus natural that the modeling of risks uses probability theory. Thus, any X of

[1] we assume that the decision maker is in a situation of risk. He knows the probability distribution P , which is exogenous, on the set of states of nature: The set (S ; $\mathcal{A}$) endowed with this probability measure is thus a probability space (S; $\mathcal{A}$;P)

V is then a random variable and has then a probability distribution denoted P_X. Let F_X[2] denote the cumulative distribution function of P_X such that $F_X(x) = P\{X \leq x\}$. Even if the distribution function F_X does not tell us what is the actual value of X, it thoroughly describes the range of possible values for X and the probabilities assigned to each of them. Let $G_X(x) = P(X > x) = 1 - F_X(x)$ be the survival function (also called tail function) and $E(X)$ the expected value of X.
For each Decision maker there exists a binary preference relation $\succeq$ (i.e. a nontrivial weak order) over V. $\succeq$ is then transitive and complete. The relation $\succeq$ is said to be "nontrivial" if there exists X and $Y \in V$ such that $X \succ Y$; "complete" if $\forall X, Y \in V$, $X \succeq Y$ or $Y \succeq X$ and "transitive" if $\forall X, Y, Z \in V$, $X \succeq Y$ and $Y \succeq Z \Rightarrow X \succeq Z$. Thus for any pair of assets X, Y; $X \succeq Y$ means that X is weakly preferred to Y by the DM, $X \succ Y$ means that X is strictly preferred to Y and $X \sim Y$ means that X and Y are considered as equivalent by the DM.

First we state three axioms which are usual and natural requirements, whatever the attitude towards risk may be.

(A.1) $\succeq$ ***respects first-order stochastic dominance***

$\forall X, Y \in V$, $[P(X \geq t) \geq P(Y \geq t)\ \forall t \in I\!R] \Rightarrow X \succeq Y$.

In words, if, for each amount t of money, the probability that lottery X yields more than t is greater than the probability that lottery Y yields more than t, then X is preferred to Y. This implies that identically distributed random variables are indifferent to the decision maker.

(A.2) Continuity with respect to monotone simple convergence

$\forall X_n, X, Y \in V$

$[X_n \downarrow X,\ X_n \succeq Y\ \forall n] \Rightarrow X \succeq Y$

[2]In words, $F_X(x)$ represents the probability that the random variable X assumes a value that is less than or equal to x.

$[X_n \uparrow X,\, X_n \preceq Y\ \forall n] \Rightarrow X \preceq Y$
$X_n \downarrow X$ (resp $X_n \uparrow X$) means that X_n is a monotonic decreasing (resp. monotonic increasing) sequence simply converging to X.

(A.3) Monotonicity

$[X \geq Y + \varepsilon.S,\, \varepsilon > 0]$ [3] $\Rightarrow X \succ Y$

One can show that any preference relation satisfying the axioms above may be characterized by a unique real number $c(X)$ to be referred to as *the certainty equivalent* of X : $X \sim c(X).S$, where $c(.)$ satisfies :

- $X \succeq Y \Leftrightarrow c(X) \geq c(Y)$.
- $X \geq Y \Rightarrow c(X) \geq c(Y)$ and $X \geq Y + \varepsilon.S, \varepsilon > 0 \Rightarrow c(X) > c(Y)$.
- $X_n, X, Y \in V$; $X_n \downarrow X \Rightarrow c(X_n) \downarrow c(X)$; $X_n \uparrow X \Rightarrow c(X_n) \uparrow c(X)$.
- $X \succeq_{FSD} Y \Rightarrow c(X) \geq c(Y)$.

Note that the existence of this certainty equivalent is guaranteed by the continuity and monotonicity assumptions, and it can be used as a representation of $\succeq$.

4 Models of decision under risk

In this paper, we aim at modeling the decision maker's preferences $(V, \succeq)$ by a real valued utility function, that is, a mapping U from V to $I\!R$ such that : $X \succeq Y \Rightarrow U(X) \geq U(Y)$. This functional will take different forms depending on the set of axioms one imposes. Let us first introduce the classical model of decision under risk, the expected utility model.

[3] For $A \in \mathcal{A}$, De Finetti's use of A to denote the characteristic function of A $[A(s) = 1$ if $s \in A$, $A(s) = 0$ if $s \notin A]$ will be adopted.

4.1 The expected utility model (EU)

The Expected Utility (EU) model, first introduced in the seminal work of von Neumann and Morgenstern (1944)[41] is the classical model of decision under risk, which furthermore satisfies the central sure thing principle of Savage,

(A.4) Sure thing principle:

Let X,$Y \in V_0$ such that $\mathcal{L}(X) = (x_1, p_1; \ldots; x_i, p_i; \ldots; x_n, p_n)$ and $\mathcal{L}(Y) = (y_1, p_1; \ldots; y_i, p_i; \ldots; y_n, p_n)$, with $x_1 \leq ... \leq x_i \leq ... \leq x_n$, $pi \geq 0$, $y_1 \leq ... \leq y_i \leq ... \leq y_n$, $pi \geq 0$ and $\sum_{i=1}^{n} p_i = 1$ and suppose that for a certain i_0, we have $x_{i_0} = y_{i_0}$.

The axiom tell us that the preference between X and Y are unchanged if we replace x_{i_0} and y_{i_0} by a common $t \in I\!R$.

In this model, preferences can be represented (see Fishburn and Wakker (1995)[26]), due both to the independence axiom and the von Neumann Morgenstern (vNM) theorem, by the expected utility denoted $E(u(X))$ such that:

$$E(u(X)) = \int_{-\infty}^{0} \left[(P(u(X) > t)) - 1\right] dt + \int_{0}^{\infty} (P(u(X) > t)dt \quad (1)$$

where u is the utility function of von Neumann Morgenstern; $u : I\!R \rightarrow I\!R$, is continuous, strictly increasing and unique up to an affine increasing transformation. The best decision being the one maximizing this Expected Utility.

For a discrete random variable $X \in V_0$ (X is a lottery) with law of probability $\mathcal{L}(X) = (x_1, p_1; \ldots; x_i, p_i; \ldots; x_n, p_n)$, with $x_1 < ... < x_i < ... < x_n$, $pi \geq 0$ and $\sum_{i=1}^{n} p_i = 1$, the formula (1) reduces to:

$$E(u(X)) = \sum_{i=1}^{n} p_i.u(x_i) \quad (2)$$

Even if the EU model has the advantage to be parsimonious (nevertheless any kind of risk aversion is characterized by a concave utility function), so many observed economic behaviors cannot be explained in the framework of this model. Consequently, we will present, next, the Rank-Dependent Expected Utility (RDU) model, a more general model, less parsimonious but more explanatory. But above all, we rapidly investigate the Dual Yaari's model which is proved to be more flexible than EU theory.

4.2 The Yaari model

One of the most successful nonexpected utility models is the dual theory of choice under risk due to Yaari (1987). In this model, **the comonotone independence axiom** will be substituted to **the sure thing principle axiom** . Let us first recall the definition of comonotonicity, a fundamental notion in the study of risk aversion.

Definition 4.1. *Yaari (1987), Schmeidler (1989):*
Two real-valued functions X and Y on S are comonotone *if for any s and $s' \in S$, $[X(s) - X(s')]\ [Y(s) - Y(s')] \geq 0$.*

Remark 4.2. *Note that comonotonicity is not a transitive relation because constant functions are comonotone with any function. Consistent with the usual conventions, random variables are said to be comonotone if they are comonotone functions almost everywhere.*

(A.4)' Comonotone Independence

$[X$ and Z are comonotone, Y and Z are comonotone, $X \sim Y] \Rightarrow X+Z \sim Y+Z$

Under axioms (A.1),(A.2),(A.3) and (A.4)', Chateauneuf (1994) showed that the function c(X) which represent preferences is not other than the certainty equivalent of Yaari(1987):

$$c(X) \ = \ \int_{-\infty}^{0} \left[f(P(X > t)) - 1 \right] dt + \int_{0}^{\infty} f(P(X > t))dt \quad (3)$$

Where $f : [0,1] \rightarrow [0,1]$ is the probability transformation function, is continuous, increasing and such that $f(0) = 0$ and $f(1) = 1$.
We can interpret the function f differently as the probability distortion or perception function since it is an adjustment of the underlying objective probability due to the subjective risk perception of the decision maker.

For a discrete random variable $X \in V_0$ with law of probability $\mathcal{L}(X) = (x_1, p_1; \ldots; x_i, p_i; \ldots; x_n, p_n)$, with $x_1 < ... < x_i < ... < x_n$, $pi \geq 0$ and $\sum_{i=1}^{n} p_i = 1$, the formula (3) reduces to:

$$c(X) \ = \ x_1 + \sum_{i=2}^{n} \left[(x_i - x_{i-1}).f(\sum_{j=i}^{n} p_j) \right] \quad (4)$$

This theory, while eliminating some of expected utility's drawbacks, shares with expected utility the completeness assumption: the decision maker must be able to rank any pair (X, Y) of lotteries.

4.3 The Rank Dependent Expected Utility model(RDU)

In order to take into account the paradoxes of Allais (1953)[7] and to separate perception of risk from the valuation of outcomes (which ones are taken into account by the same tool, the utility function in EU theory) an alternative theory the rank dependent expected utility(RDU) first elaborated by Quiggin (1982)[33] under the denomination of "Anticipated Utility" has been developed since the early eighties. The Rank Dependent Expected Utility model

is the most widely used, and arguably the most empirically successful, generalization of the expected utility model(EU). Variants of this model are due to Yaari(1987)[44] and Allais (1988)[8]. More general axiomatizations can be found in Wakker (1994)[42], Chateauneuf (1999)[12].
Let us recall that a RDU DM weakly prefers$X$ to Y, $X, Y \in V$ if and only if $E(u(X)) \geq E(u(Y))$, where $E(u(Z))$ is defined for every $Z \in V$ by :

$$E(u(Z)) \;=\; \int_{-\infty}^{0} [f(P(u(Z) > t)) - 1]\, dt + \int_{0}^{\infty} f(P(u(Z) > t))dt \quad (5)$$

Roughly speaking, in RDU theory, individuals' preferences over risky prospects are represented by the mathematical expectation of a utility function u with respect to a transformation f of the outcomes cumulative probabilities. u utility of wealth, $u : I\!R \to I\!R$ is assumed to be cardinal (i.e., defined up to a positive affine transformation), strictly increasing and continuous.
$f : [0, 1] \to [0, 1]$ the probability transformation function (as in Yaari) is assumed to be strictly increasing, continuous and such that $f(0) = 0$ and $f(1) = 1$. Note that, in this model, the transformation function f is defined under cumulative probabilities rather than simple probabilities. That's why the ranking of outcomes is fundamental (which explains the denomination of Rank Dependent Expected Utility).
For a discrete random variable Z with probability law
$\mathcal{L}(Z) = (z_1, p_1; \ldots; z_k, p_k; \ldots; z_n, p_n)$, where $z_1 \leq z_2 \leq \cdots \leq z_n$, $p_i \geq 0$ and $\sum_{i=1}^{n} p_i = 1$, the formula (5) reduces to :

$$E(u(Z)) \;=\; u(z_1) + \sum_{i=2}^{n}(u(z_i) - u(z_{i-1})) \left[f(\sum_{j=i}^{n} p_j) \right] \quad (6)$$

Such a formula is meaningful : the DM takes for sure the utility of the minimum payoff, and then add the successive possible additional increments of utility weighted by his personal perception of the related probability.
RDU theory reduces to EU theory if f is the identity function, and RDU theory reduces to the dual theory of Yaari if u is the identity function. Unlike EU, RDU preferences allows us to discriminate amongst different notions of risk aversion.

5 Definition of different notions of risk aversion and relationships in model-free

5.1 Definition of risk aversion

The most natural way to define risk aversion is as a tendency to choose, when possible, to avoid risk. More precisely, a weak risk averse decision-maker always prefers to any random variable X the certainty of its expected value E(X). Differently, we can define risk aversion as a dislike of some type of (mean preserving) increasing risk. For example, following Rothschild and Stiglitz (1970)[38], strong risk aversion refers to an aversion towards mean preserving spreads . While risk aversion is defined by decision theorists as a preference for a sure outcome over a chance prospect with equal or greater expected value, risk seeking, in contrast, is defined as a preference for a chance prospect over a sure outcome of equal or greater expected value.
We are now to give definitions for the comparison of various probability distributions. They are sometimes called stochastic orders. One can detect that stochastic dominance is a form of stochastic ordering. The term is used in decision theory to refer to situations where one lottery (a probability distribution over outcomes) can be ranked as superior to another. It is based on preferences regarding outcomes (e.g., if each outcome is expressed as a

number, gain or utility, a higher value is preferred), but requires only limited knowledge of preferences with regard to distributions of outcomes, which depend on risk aversion. Indeed, the canonical case of stochastic dominance is referred to as first-order stochastic dominance, defined as follows:

Definition 5.1. ***First order stochastic dominance*** [4]

Let X and Y the elements of V, X is said to dominate Y for the first order stochastic dominance to be denoted ($X \succeq_{FSD} Y$) if :

$$Pr[X > t] \geq Pr[Y > t] \quad \forall t \in \mathbb{R}$$

$$i.e, \quad F_X(t) \leq F_Y(t) \quad \forall t \in \mathbb{R}$$

For all t, the probability of having more than t is always larger for X than for Y. For example, consider a coin-toss where heads and tails give returns 1 and 3 respectively for A, and 2 and 1 respectively for B. In this example, clearly A has first-order stochastic dominance over B. Further, although when A dominates B, the expected value of the payoff under A will be greater than the expected value of the payoff under B, this is not a sufficient condition, and so one cannot order lotteries by comparing the means of their probability distributions.

The second concept is weaker than FSD and is called second-degree stochastic dominance (SSD). This holds whenever one distribution is equal to or larger than, that under the other cumulative distribution.

Definition 5.2. ***Second order stochastic dominance [Rothschild and Stiglitz, 1970[38]]***

[4] Stochastic dominance is a term which refers to a set of relations that may hold between a pair of distribution

Let X, $Y \in V$, X is said to dominate Y for the second order stochastic dominance to be denoted ($X \succeq_{SSD} Y$) if :

$$\int_{-\infty}^{x} F_X(t)dt \leq \int_{-\infty}^{x} F_Y(t)dt \quad \forall x \in \mathbb{R}.$$

The definition of SSD gives the following implications:

1. If X first order stochastically dominates Y , then X second-order stochastically dominates Y.

2. If X second-order stochastically dominates Y, then $E(X) \geq E(Y)$ but the reverse is not necessarily true.

Since SSD is stronger than FSD, a larger set of distributions can be ordered under SSD. Since the concavity assumption is often used as a necessary condition for the existence of a maximum or because it implies risk-aversion, the SSD condition might be the more important one for a large set of fields. Obviously, any result within the framework of risk-aversion can be established directly by means of SSD. Conversely, any case of preference under risk aversion must imply SSD.

For a DM with a preference relation $\succeq$ on V , we now give some model-free concepts of risk aversion and their precise definitions. Thus, the definitions below are given independently of any model.
The first notion of risk aversion corresponds to a propensity to choose, when possible, to avoid risk, or more precisely, to always prefer to any random variable X the certainty of its expected value $E(X)$.

Definition 5.3. ***Weak Risk Aversion [Arrow(1965)[10], Pratt(1964)[32]]***

A DM exhibits Weak Risk Aversion (WRA) if, for any random variable X of V, he prefers to the random variable X, its expected value $E(X)$ with certainty:

$$\forall X \in V, \quad E(X) \succeq X$$

has Weak Risk-Seeking (WRS) if $\forall X \in V,\ X \succeq E(X)$;
is risk-neutral if $\forall X \in V,\ X \sim E(X)$.

To introduce the notion of strong risk aversion, we have first to define Mean Preserving Spread based on second order stochastic dominance already defined.

Definition 5.4. ***Mean Preserving Spread [Rothschild and Stiglitz (1970)[38]]***

For X and Y with the same mean, Y is a general mean preserving increase in risk or Mean Preserving Spread (MPS) of X if :

$$\int_{-\infty}^{x} F_X(t)dt \leq \int_{-\infty}^{x} F_Y(t)dt \ ,\forall x \in \mathbb{R}$$

$$E(X) = E(Y) \ \ and \ \ X \ \ SSD \ \ Y \quad \Rightarrow \quad Y \ \ MPS \ \ X$$

Remark 5.5. *This notion of increase in risk is considered as a special case, for equal means, of second order stochastic dominance and it could be explained by the fact that the more risky* Y *is obtained by adding a noise* Z[5] *to* X.

Definition 5.6. ***Strong Risk Aversion [Hadar and Russell (1969)[28], Rothschild and Stiglitz (1970)[38]]***

A DM exhibits Strong Risk Aversion (SRA) if for any pair of random variables $X, Y \in V$ *with* Y *being a Mean Preserving Spread of* X*, he always*

[5] $E(Z \mid X) = 0$

prefers X to Y :

$$\forall X, Y \in V, \;\; YMPSX \Rightarrow X \succeq Y$$

has Strong Risk-Seeking (SRS) if $\forall X, Y \in V$*,* Y *MPS* $X \Rightarrow Y \succeq X$*;*
is risk-neutral if $X \sim Y$*.*

While weak risk aversion can be viewed as aversion to risk, strong risk aversion can be viewed as aversion to any increase in risk. Intuitively, these two notions capture distinct behaviors: a DM may want to avoid completely risk when possible, but when he cannot do so and has to choose between two situations where he cannot avoid risk, he could choose the riskier one, hoping to get the best consequences. However, the notion of strong risk aversion is always considered as too strong by some DM. This is why Quiggin(1991) proposed the weaker notion called monotone risk aversion based on comonotonicity.

Definition 5.7. ***Mean Preserving Monotone Spread [Rothschild and Stiglitz (1970)[38], Quiggin (1991)[34]]***
For $X, Y \in V$, the distribution of Y is a monotone increase in risk *of the distribution of X, or, Y is a* mean preserving monotone spread *of X, if $\exists \;\; Z \in V$ such that $E(Z) = 0$, Z and X are comonotone and $Y =_d$ [6] $X + Z$. Thus, X is said to be less risky than Y for the monotone risk order denoted $X \succeq_M Y$.*

Indeed, this definition is meaningful since adding such a Z to X maintains a constant mean, but in a very intuitive way increases monotonously the risk. The following concept is based on aversion to monotone increases in risk.

[6] Y has the same probability distribution than $X + Z$

Definition 5.8. ***Monotone Risk Aversion [Quiggin (1991)[34]]***

A DM is monotone risk averse if for any $X, Y \in V$ with equal means such that Y is a monotone mean preserving spread of X, the DM weakly prefers X to Y.

$$i.e, \forall X, Y \in V, \quad X \succeq_M Y \Rightarrow X \succeq Y$$

Abouda and Chateauneuf (2002)[3], Abouda (2008)[1] have introduced two new concepts of risk aversion namely symmetrical monotone risk aversion and preference for perfect hedging (or alternately attraction for certainty).

Definition 5.9. ***Symmetrical Monotone Risk Order [Abouda and Chateauneuf 2002[4]]***

Let $X, Y \in V$, X is less risky than Y for the symmetrical monotone risk order denoted $X \succeq_{SM} Y$, if there exists $Z \in V$ such that $E(Z) = 0$, Z comonotone with X, $Z =_d -Z$ and $Y =_d X + Z$.

Definition 5.10. ***Symmetrical Monotone Risk Aversion [Abouda and Chateauneuf 2002[4]]***

A DM is said to be symmetrical monotone risk averse denoted SMRA if:

$$\forall X, Y \in V, \quad X \succeq_{SM} Y \Rightarrow X \succeq Y$$

Definition 5.11. ***Preference For Perfect Hedging [Abouda and Chateauneuf 2002[3], Abouda 2008[1]]***

The definition of preference for perfect hedging can take one of the three following assertions:

(i) $[X, Y \in V, \alpha \in [0,1], \alpha X + (1-\alpha)Y = a.S, a \in \mathbb{R}] \Rightarrow a.S \succeq X$ *or* Y.

(ii) $[X, Y \in V, X \succeq Y, \alpha \in [0,1], \alpha X + (1-\alpha)Y = a.S, a \in \mathbb{R}] \Rightarrow a.S \succeq Y$.

(iii) $[X, Y \in V, X \sim Y, \alpha \in [0,1], \alpha X + (1-\alpha)Y = a.S, a \in \mathbb{R}] \Rightarrow a.S \succeq Y$.

Remark 5.12. *Preference for perfect hedging means that if the decision maker can attain certainty by a convex combination of two assets, then he prefers certainty to one of these assets.*

Chateauneuf and Tallon (2002)[20], Chateauneuf and Lakhnati (2007)[19] have introduced a generalization of preference for perfect hedging which is called preference for sure diversification.

Definition 5.13. ***Preference For Sure Diversification (Chateauneuf and Tallon (2002)[20])***

$\succeq$ *exhibits preference for sure diversification if for any* $X_1, ..., X_n \in V$ *;* $\alpha_1, ..., \alpha_n \geq 0$ *such that* $\sum_{i=1}^{n} \alpha_i = 1$ *and* $a \in I\!R$

$$[X_1 \sim X_2 \sim ... \sim X_n \ \ \textit{and} \ \ \sum_{i=1}^{n} \alpha_i X_i = a] \Rightarrow a \succeq X_i, \quad \forall i$$

.

Remark 5.14. *Preference for sure diversification means that if the decision maker can attain certainty by a convex combination of equally desirable assets, then he prefers certainty to any of these assets.*

Now, we introduce the concept of convex preferences since it will be useful in the sequel. To start, let us underline that convexity plays a crucial role in proving the existence of various equilibria in cooperative and noncooperative game theories. While convex analysis on vector spaces has brought a plenty of fruitful results to optimization theory and its application to economics and game theory, it is apparent that standard convex analysis is inadequate to deal with topological spaces which lack a vector space structure. Thus, one could get away with the following definition.

Definition 5.15. ***Convex preferences***

A preference is said to be convex if:

$$\forall X, Y \in V \quad and \quad \alpha \in [0,1], \quad X \succeq Y \quad \Rightarrow \quad \alpha X + (1-\alpha)Y \succeq Y.$$

Remark 5.16. *convex preferences means that convex combination of two assets is preferred to one of these assets.*

By proposition **5.17** below, we will define convex preferences indifferently by (i) or (ii) or (iii) and remark **5.18** gives the relation between preference for perfect hedging and convex preferences.

Proposition 5.17. *Convex preference can be defined by (i), (ii) or (iii) :*
(i) $\forall X, Y \in V$ *and* $\alpha \in [0,1], \alpha X + (1-\alpha)Y \succeq X$ *or* Y.
(ii) $\forall X, Y \in V$ *and* $\alpha \in [0,1]$, $X \succeq Y \Rightarrow \alpha X + (1-\alpha)Y \succeq Y$.
(iii) $\forall X, Y \in V$ *and* $\alpha \in [0,1]$, $X \sim Y \Rightarrow \alpha X + (1-\alpha)Y \succeq Y$.

Proof.

- It is clear that $(i) \Rightarrow (ii) \Rightarrow (iii)$

- $(iii) \Rightarrow (i)$

Let X, $Y \in V$ and $\alpha \in [0,1]$
We can suppose that $X \succeq Y$ (either-wise we interchange X and Y).
Let $c \leq 0$ / $X + c \sim Y$. Than by (iii) we have $\alpha(X+c) + (1-\alpha)Y \succeq Y$ than $\alpha X + (1-\alpha)Y \succeq Y$. (Because $\alpha c \leq 0$) □

Remark 5.18. *we see obviously that Preference for perfect hedging is weaker than convex preferences.*

Let us now introduce a new concept of risk aversion that we call weak weak risk aversion. Property **5.32** proves that weak weak risk aversion is weaker than preference for perfect hedging.

Definition 5.19. ***Weak weak risk aversion***

A decision-maker is weak weakly risk averse if :

$$\forall X \in V,\ E(X) \succeq X \ \ or \ \ 2E(X) - X.$$

Abouda, M and Farhoud, E. (2010)[5] introduced a new concept of risk aversion namely anti-monotone risk aversion related to the concept of anti-comonotony previously studied in Abouda, M. Aouani, Z. and Chateauneuf, A. (2008)[2]. Thus, let us first introduce this latter concept.

Definition 5.20. ***Strict anti-comonotony (Abouda, M and Farhoud, E. (2010)[5])***

Two real-valued functions X and Y on S are strictly anti-comonotone *if for any s and $s' \in S$,*

$$X(s) > X(s') \Longrightarrow Y(s) < Y(s')$$

and

$$X(s) = X(s') \Longrightarrow Y(s) = Y(s').$$

We can define differently strict anti-comonotony as follows:

Definition 5.21. ***Strict anti-comonotony (Abouda, M and Farhoud, E. (2010)[5])***

Two real-valued functions X and Y on S are strictly anti-comonotone *if for* *any s and $s' \in S$,*

$$X(s) > X(s') \iff Y(s) < Y(s').$$

Definition 5.22. ***Anti-monotone risk order (Abouda, M and Farhoud, E. (2010)[5])***
Let $X, Y \in V$, X is more risky than Y for the anti-monotone risk order denoted $Y \succeq_{AM} X$, if X and Y are comonotone and there exists $Z \in V$ strictly anti-comonotone with X such that $E(Z) = 0$ and $Y =_d X + Z$.

Definition 5.23. ***Anti-monotone risk aversion (Abouda, M and Farhoud, E. (2010)[5])***
A DM is anti-monotone risk averse denoted ARA if:

$$\forall \;\; X, Y \in V, \;\; X \succeq_{AM} Y \implies X \succeq Y.$$

5.2 Risk aversion and Relationships in model-free

This section is reserved to the study of the relationships that can exist among different notions of risk aversions already mentioned above. For generality, implications should be proved in a model-free (i.e, independently of any representation). Several authors - Yaari (1969)[43], Cohen (1995)[21], Chateauneuf (1999)[12], Grant and Quiggin (2001), Chateauneuf, Cohen and Meilijson (2004; 2005), Mathew J. Ryan (2006)[31], Chateauneuf and Lakhnati (2007)[19], Abouda (2008)[1], Aouani and Chateauneuf (2008)[6], Abouda, Aouani and Chateauneuf (2010)[2], Abouda and Farhoud (2010)[5]- have studied the relationships among different notions of such behaviors in particular models (EU, RDU, ...) or independently of any representation which is the focus of our present paper. In the following, we will start by presenting the relationship between weak risk aversion and preference for perfect hedging (or, equivalently attraction for certainty) as shown in Abouda(2008) where he proved that preference for perfect hedging is weaker than weak risk aversion.

5.2.1 Preference for perfect hedging and weak risk aversion

Theorem **5.24** below gives the relation between weak risk aversion and attraction for perfect hedging.

Theorem 5.24.

$$Weak\ \ risk\ \ aversion \implies Preference\ \ for\ \ perfect\ \ hedging.$$

Proof.

Let $X \sim Y$ and α / $\alpha X + (1-\alpha)Y = a.S$

By hypothesis $E(X).S \succeq X$ and $E(Y).S \succeq Y$ then

$$min(E(X), E(Y)).S \succeq Y \tag{1}$$

We have $a = \alpha E(X) + (1-\alpha)E(Y) \geq min(E(X), E(Y))$ then

$$a.S \succeq min(E(X), E(Y)).S \tag{2}$$

(1) and (2) $\Rightarrow a.S \succeq Y$, hence (iii) of definition 5.1.7 is satisfied.

□

Remark 5.25. *Note that the converse of theorem **5.24** is false. In Yaari's model for example Abouda and Chateauneuf (2002)[3] have shown that we have preference for perfect hedging if and only if* $f(p) + f(1-p) \leq 1,\ \forall p \in [0,1]$ *which is a risk aversion weaker than the weak risk aversion characterized in this model by* $f(p) \leq p,\ \forall p \in [0,1]$.

5.2.2 Preference for sure diversification and weak risk aversion

Chateauneuf and lakhnati [19] proved that preference for sure diversification is equivalent to weak risk aversion independently of any model.

Theorem 5.26. ***Chateauneuf and Lakhnati (2007)[19]***
For a DM, with compact continuous and monotone preference , the following two assertions are equivalent:

(i) $\succeq$ *exhibits preference for sure diversification .*

(ii) The DM is weakly risk averse.

For more details, see proof in Chateaneuf and Lakhnati (2007)[19]

5.2.3 Symmetrical Monotone risk aversion and Monotone risk aversion

One can detect that symmetrical monotone risk aversion is a particular case of monotone risk aversion.

Theorem 5.27. ***Abouda and Chateauneuf (2002)[4]***

$$Monotone\ risk\ aversion \implies Symmetrical\ monotone\ risk\ aversion.$$

Symmetrical monotone risk aversion is then weaker than monotone risk aversion.

5.2.4 Strong risk aversion, Monotone risk aversion and weak risk aversion

Theorem 5.28.

$$Strong\ \ risk\ \ aversion \implies Weak\ \ risk\ \ aversion.$$

Proof. let $X \in V$;

Since X MPS $E(X)$ (see Yaari 1969[43]),

Definition **5.6** gives $E(X) \succeq X$.

□

Note that the reciprocal implication is not true in general.
In what follows, we will focus on the connection between strong, monotone and weak risk aversion. To do so, the remainder will be divided into two separated parts: The first one deals with the relationship between monotone and weak risk aversion where we show that monotone risk aversion implies weak risk aversion in a simple and short way.

Theorem 5.29.

$$Monotone\ \ risk\ \ aversion \implies Weak\ \ risk\ \ aversion.$$

Proof. Let $X \in V$

Let $Z = X - E(X)$

Z and $E(X)$ are comonotone since $E(X)$ is constant, and

$E(Z) = E[X - E(X)] = 0$

Definition **5.7** gives $E(X) \succeq_M E(X) + Z$.

Given that $X = E(X) + Z$ and according to the definition **5.8**, one can obtain $X \preceq E(X)$.

□

Remark 5.30. *To study the relation that ties strong and monotone risk aversion, we must first understand the kind of connection that exists between two fundamental notions of increase in risk namely mean preserving spread (MPS) and monotone mean preserving spread (MMPS). By remark **5.5**, Y is a mean preserving spread of X if and only if there exists a random variable Z such that $Y =_d X + Z$ and $E(Z \mid X) = 0$ (see Rothschild and Stiglitz (1979)[38]). Similarly, the mean preserving monotone spread requires that there exists a random variable Z, with mean zero, such that Z and X are comonotone. Thus, monotone mean preserving spread implies mean preserving spread. The latter result is considered as one of the properties related to monotone mean preserving spread using simple mean preserving spread [7] (see Quiggin (1992)[35]).*

Theorem 5.31.

$$Strong\ \ risk\ \ aversion \implies Monotone\ \ risk\ \ aversion.$$

Proof. Let $X, Y \in V$ such that Y MMPS X;
Following remark **5.30**, we have Y MPS X.
By definition **5.6**, one can get $X \succeq Y$

□

5.2.5 Preference for perfect hedging and weak weak risk aversion

Let us now prove that Preference for perfect hedging is stronger than weak weak risk aversion.

[7] A simple mean preserving spread is easily proved to be a mean preserving spread.

Theorem 5.32.

$$Preference \;\; for \;\; perfect \;\; hedging \implies Weak \;\; weak \;\; risk \;\; averse.$$

Proof.

Let $X \in V$. We have $\frac{1}{2}X + \frac{1}{2}(2E(X) - X) = E(X)$

Then, by (i) of definition **5.11**, we have $E(X) \succeq X$ or $2E(X) - X$. □

5.2.6 Anti-monotone risk aversion, weak risk aversion and monotone risk aversion

This subsection is reserved to the study of the relationship between anti-monotone, monotone and weak risk aversion. Theorem **5.33** and **5.34** below show that anti-monotone risk aversion is weaker than monotone risk aversion while stronger than weak risk aversion.

Theorem 5.33. ***(Abouda, M and Farhoud, E. (2010)[5])***

$$Monotone \;\; risk \;\; aversion \implies Anti-monotone \;\; risk \;\; aversion.$$

Proof.

Let $X, Z \in V$ such that Z strict anti-comonotone with X, $E(Z) = 0$ and $X + Z$ comonotone with X.

Given that $-Z$ is strict anti-comonotone with Z then, we have $-Z$ comonotone with $X + Z$.

Definition **5.7** gives $X + Z \succeq_M X + Z + (-Z)$.

Then, according to definition **5.8**, we have $X + Z \succeq X$. □

Theorem 5.34. ***(Abouda, M and Farhoud, E. (2010)[5])***

$$Anti-monotone \;\; risk \;\; aversion \implies Weak \;\; risk \;\; averse.$$

Proof.

Let $Z = E(X) - X$

We can see easily that Z is strict anti-comonotone with X, $E(Z) = 0$ and $X + Z$ comonotone with X.

Then, by definition **5.22**, one can obtain $X + Z \succeq_{AM} X$,

Then, following our hypothesis and definition **5.23**, we have $E(X) \succeq X$.

$\square$

Finally, to conclude this section, theorem **5.35** summarizes the different model-free relationships among risk aversion that we have met during this work. Thus, the classification goes from the strongest one to the weakest one as shown below.

Theorem 5.35.

Strong Risk Aversion

$\Downarrow$

Monotone Risk Aversion

$\Downarrow$

Anti-monotone risk aversion

$\Downarrow$

Weak Risk Aversion

$\Downarrow$

preference for perfect hedging

$\Downarrow$

weak weak risk aversion

while the reciprocal assertions are not necessarily true.

6 Characterization of different risk aversion in models of decision under risk

Following Rothschild and Stiglitz (1970)[38], strong risk aversion is characterized, in the EU model, by the concavity of u. Due to the Jensen inequality, weak risk aversion is also characterized by a concave utility function within the framework of expected utility. Monotone risk aversion, being a halfway between the two previous aversion, will be then characterized by u concave. Similarly, anti-monotone risk aversion is between monotone and weak risk aversion then it is characterized also by u concave. Thus, it's clear that aversion to risk in the EU model is characterized by the concavity of the utility function (see Rothschild and Stiglitz (1970)[38]). Hence in EU model, $\succeq$ exhibits preference for sure diversification if and only if u is concave(see Chateauneuf and Lakhnati (2007)[19]).Thus, in the EU model, it is impossible to discriminate amongst different forms of behaviors under risk. Thus, any DM who is weakly risk averse but not strongly risk averse cannot satisfy the model's axioms. However, this model shows a lack of flexibility and explanatory power in terms of modeling the decision maker'behavior towards risk.

By contrast, in the Yaari's model, strong risk aversion is characterized by the convexity of the probability weighting function f (see Roell (1985)[37], Yaari (1987)[44] and Chateauneuf (1991)[13]) while monotone and weak risk aversion are characterized by $f(p) \leq p$ (see Yaari (1987)[44], Quiggin (1992)[35], Chateauneuf and Cohen (1994)[14]). Given that anti-monotone risk aversion is a halfway between monotone and weak risk aversion, it will be characterized also by $f(p) \leq p$ in the yaari's model. Symmetrical monotone risk aversion will be characterized by $f(p) + f(1-p) \leq 1$ (see Abouda and Chateauneuf (2002)[4]). Moreover, it's straightforward that $f(p) \leq p$ implies $f(p) + f(1-p) \leq 1$. As a consequence, $SMRA$ still always weaker than the standard weak risk aversion while equivalent with preference for perfect hedging.

Remark 6.1. *Even if the Yaari's model was born to eliminate some of the drawbacks linked to the EU model (all risk aversion are considered as equivalents), openly, the problem had not been totaly solved and we see ,yet, that monotone and weak risk aversion, for example, still always equivalents in the framework of Yaari. Hence the necessity of a model that is more flexible and has more explanatory power.*

Contrary to the EU theory where all different concepts of risk aversion are considered as equivalents (u concave), the RDU model is built, in general, to bring new tools that allow us to distinguish easily between these behaviors through their different characterizations for both u and f. It's well known on the one hand, that a RDU DM is strongly risk averse if and only if u is concave and f is convex (see Chew, Karni and Safra (1987)[23], Machina (1982)[30]). On the other hand, symmetrical monotone risk aversion ($SMRA$) is characterized, in the RDU model, by a comparison of an index of pessimism linked to the probability transformation function f $(P_f = \inf_{p \in]0, \frac{1}{2}]} \frac{(1-f(1-p))}{f(p)})$ with an index of greediness linked to the utility function $u(G_u = sup_{y \leq x} \frac{u'(x)}{u'(y)})$ (see Abouda and Chateauneuf (2002)[4]) quite as it was done by Chateauneuf, Cohen and Meilijson (2005)[] to characterize monotone risk aversion (MRA). Note that, according to Abouda and Chateauneuf (2002)[4], a RDU DM is risk averse in the sense of $\succeq_{SM}$ if and only if he is more pessimistic than greedy. A similar result has been earlier obtained in Chateauneuf, Cohen, and Meilijson (2005)[16] for the case of monotone risk aversion (i.e, A DM is Monotone Risk Averse (MRA) if and only if his index of pessimism under risk exceeds his index of greediness (see also Chateauneuf, Cohen and Melijson (2005)[16]). Thus, $SMRA$ and MRA are considered as equivalents in the framework of RDU. Note that any type of risk aversion in RDU implies $f(p) + f(1-p) < 1$[15]. Many authors have tried to find a clear-cut characterization of weak risk averse decision makers in the framework of RDU model. Chateauneuf and Cohen (1994)[14] have succeed to find a necessary condition for weak risk aversion namely $f(p) \leq p$ and a sufficient one in order to characterize weak risk aversion in the RDU model. According to Chateauneuf and Cohen (1994)[14]: For a RDU DM with a differentiable and increasing u and an increasing probability weighting function f, if $\exists k \geq 1$ that satisfies the following conditions:

(i) $u'(x) \leq k \ \frac{u(x)-u(y)}{x-y}, \ y < x$ and

(ii)f is such that $f(p) \leq p^k$, $\forall p \in [0;1]$

Then, our DM is weakly risk averse. Since preference for sure diversification is equivalent to weak risk aversion, we can say that they have the same characterization as mentioned in the table below.

Remark 6.2. *Note that defining* $G_u = sup_{y \leq x} \frac{u'(x)}{u'(y)}$ *and* $P_f = \inf_{p \in]0,1]} \frac{\frac{(1-f(p))}{f(p)}}{\frac{f(p)}{p}}$ *as respectively an index of greediness (also called index of non concavity) and an index of pessimism. Let us underline that the former satisfies* $G_u \geq 1$ *and the value* 1 *is equivalent to the concavity of the utility function* u*. Similariy, the latter satisfies* $P_f \geq 1$ *as soon as* $f(p) \leq p$.

Hereafter, we intend to build a recapitulative table that summarizes the characterization of the different forms of behavior under risk relatively to the models already mentioned above.

	EU	Yaari	RDU
SRA	u concave	f convex	f convex and u concave
MRA	u concave	$f(p) \leq p \;\; \forall p \in [0,1]$	$\inf_{p\in]0,1]} \frac{\frac{(1-f(p))}{f(p)}}{\frac{f(p)}{p}} \geq sup_{y\leq x} \frac{u'(x)}{u'(y)}$
ARA	u concave	$f(p) \leq p \;\; \forall p \in [0,1]$	?
WRA	u concave	$f(p) \leq p \;\; \forall p \in [0,1]$	$\exists k$ s.t $u'(x) \leq k \;\; \frac{u(x)-u(y)}{x-y}, y < x$
Pref for sure div	u concave	$f(p) \leq p \;\; \forall p \in [0,1]$	$\exists k$ s.t $u'(x) \leq k \;\; \frac{u(x)-u(y)}{x-y}, y < x$
SMRA	u concave	$f(p) + f(1-p) \leq 1$	$\inf_{p\in]0,\frac{1}{2}]} \frac{(1-f(1-p))}{f(p)} \geq sup_{y\leq x} \frac{u'(x)}{u'(y)}$
Pref for perf hedg	u concave	$f(p) + f(1-p) \leq 1$	?

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